NASA TECHNICAL NOTE

NASA TN D-3128

c. 1

NASA TN D-3128

ON THE HIGH ORDER EFFECTS IN THE METHODS OF KRYLOV-BOGOLIUBOV AND POINCARÉ

by Peter Musen

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JANUARY 1966

ON THE HIGH ORDER EFFECTS IN THE METHODS OF KRYLOV-BOGOLIUBOV AND POINCARÉ

By Peter Musen

Goddard Space Flight Center
Greenbelt, Md.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $1.00

ABSTRACT

In this article the formal process for determining the higher order perturbations of the orbital elements is developed by using the methods of Krylov-Bogoliubov and Poincaré. Such a development is necessary, for example, in the lunar problem where very high order perturbations have to be determined. The differential equations are formed for the elements which are affected only by the long period and the secular terms. The problem of determining these elements, as well as eliminating the short period effects, is reduced to solving a set of partial differential equations, step by step. By developing the displacement operator into a series of the differential operators of Faa de Bruno we can write these equations in a concise form.

CONTENTS

ON THE HIGH ORDER EFFECTS IN THE METHODS OF KRYLOV-BOGOLIUBOV AND POINCARÉ

by
Peter Musen
Goddard Space Flight Center

INTRODUCTION

In this article we develop the formalism for determining the general perturbations of higher orders in celestial mechanics by the methods of Krylov-Bogoliubov (1961) and of Poincaré (1892) and von Zeipel (1916). The solution is obtained in terms of the Krylov-Bogoliubov averaging operator, of Faa de Bruno (1855) differential operators, and of the integrating operator.

In the method of Krylov-Bogoliubov, as in that of Poincaré, the final goal is to eliminate the short period effects and derive the elements affected only by the long period and secular perturbations. The original work of Krylov and Bogoliubov was influenced by the problems of celestial mechanics. A close look at Le Verrier's (1856) method for the general perturbations, reveals the same basic idea, but the method of Krylov and Bogoliubov achieved fame under its present name because of its extensive application to other problems of theoretical physics. In most such problems there is no need to compute the effects of higher orders: Normally only the effects of the first and second orders, rarely of the third order, are computed. The standard presentation of the method does not go beyond these limits. However, this accuracy is insufficient from the standpoint of celestial mechanics. In the lunar problem, to secure the necessary accuracy of the long period terms, we must go up to the ninth order with respect to the ratio of mean motions of the satellite and of the sun.

Thus, the formalism of the Krylov-Bogoliubov method must be extended to cover such cases and especially to provide for the determination of the long period effects of higher orders. The long period and the secular effects are chiefly responsible for the behavior and stability of the orbits of the celestial body, and their accurate determination is of great importance. The positive characteristic of the method of Krylov and Bogoliubov is that the canonical form of the equations of motion is not required, and thus the method can be applied to a much wider range of problems than the method of Poincaré. However, the number of partial differential equations to be solved in the process of eliminating the short period terms increases, as compared to the method of Poincaré. It is the price paid for extending the domain of applicability.

In the method of Poincaré the equations of motion have the canonical form, and the problem of eliminating the short period terms from the coordinates and momenta reduces to eliminating such terms from the Hamiltonian by means of an appropriate canonical transformation. Assuming that the characteristic function S of this transformation is developable into a power series with respect to a small parameter, we reduce the determination of S to the solution of a chain of partial differential equations, step by step. Recently, Giacaglia (1964) has established the general form of these equations.

We show in this work that the partial differential equations of the method of Poincaré take an especially concise form if written in terms of Faa de Bruno (1855) operators.

HIGHER ORDER PER PERTURBATIONS IN THE KRYLOV-BOGOLIUBOV METHOD

Consider the system of vectorial differential equations

$$\frac{dx}{dt} = X(x; y, \eta), \tag{1}$$

$$\frac{dy}{dt} = \lambda(x) + Y(x; y, \eta), \tag{2}$$

$$\frac{d\eta}{dt} = H(x; y, \eta), \tag{3}$$

where X, Y, H are periodic in vectors y and η with the period 2π in each component. These vectors are assumed to be developable in powers of a small parameter. We have

$$X = \sum_{j=1}^{\infty} X_j\,(x; y, \eta), \tag{4}$$

$$Y = \sum_{j=1}^{\infty} Y_j\,(x; y, \eta), \tag{5}$$

$$H = \sum_{j=1}^{\infty} H_j\,(x; y, \eta), \tag{6}$$

where the functions X_j, Y_j, H_j are of the form

$$F = \sum_{n,\nu} F_{n,\nu}(x) \exp i\,(n \cdot y + \nu \cdot \eta), \tag{7}$$

where n and ν are vectors whose components are integers.

The terms in (7) are:

the short periodic if $n \neq 0$,

the long periodic if $n = 0$ but $\nu \neq 0$;

the secular if $n = 0$ and $\nu = 0$.

The averaging operator M extracts the long period and the secular terms from (7). Thus

$$MF = \sum_{\nu} F_{0,\nu}(x) \exp i\, \nu \cdot \eta$$

In addition to the Krylov-Bogoliubov operator M, it is also convenient to use the operator P which extracts the short period terms only:

$$PF = \sum_{n \neq 0,\nu} F_{n,\nu}(x) \exp i\,(n \cdot y + \nu \cdot \eta)$$

In the further exposition we make use of the partial del-operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial \eta$ and introduce a partial differential equation of the form

$$\lambda(x) \cdot \frac{\partial \psi}{\partial y} = PF.$$

Evidently

$$\psi = \sum_{n \neq 0,\nu} \frac{F_{n,\nu}}{\lambda \cdot n} \exp i(n \cdot y + \nu \cdot \eta)$$

Introducing the integrating operator Q we can write

$$\psi = QPF.$$

Let us determine the transformation

$$x = x^* + a(x^*; y^*, \eta^*), \tag{8}$$

$$y = y^* + b(x^*; y^*, \eta^*), \tag{9}$$

$$\eta = \eta^* + \beta(x^*; y^*, \eta^*) \tag{10}$$

in such a way that the differential equations for the new variables

$$\frac{dx^*}{dt} = X^*(x^*; -, \eta^*), \tag{11}$$

$$\frac{dy^*}{dt} = \lambda(x^*) + Y^*(x^*; -, \eta^*), \tag{12}$$

$$\frac{d\eta^*}{dt} = H^*(x^*; -, \eta^*) \tag{13}$$

do not contain the new short period argument y^*; we put a dash in place of y^* to emphasize its absence.

We shall determine the formal developments

$$a = \sum_{j=1}^{\infty} a_j, \quad b = \sum_{j=1}^{\infty} \beta = \sum_{j=1}^{\infty} \beta_j, \tag{14}$$

$$X^* = \sum_{j=1}^{\infty} X_j^*, \quad Y^* = \sum_{j=1}^{\infty} Y_j^*, \quad H^* = \sum_{j=1}^{\infty} H_j^* \tag{15}$$

in such a way that (11) - (13) have the prescribed form. It follows from these equations that the operator d/dt can be written in the form

$$\frac{d}{dt} = \lambda(x^*) \cdot \frac{\partial}{\partial y^*} + D,$$

where

$$D = X^* \cdot \frac{\partial}{\partial x^*} + Y^* \cdot \frac{\partial}{\partial y^*} + H^* \cdot \frac{\partial}{\partial \eta^*}$$

From (8) - (10) and (11) - (13) we have

$$\frac{dx}{dt} = X^* + \lambda \cdot \frac{\partial a}{\partial y^*} + Da, \tag{16}$$

$$\frac{dy}{dt} = \lambda + Y^* + \lambda \cdot \frac{\partial b}{\partial y^*} + Db, \tag{17}$$

$$\frac{d\eta}{dt} = H^* + \lambda \cdot \frac{\partial \beta}{\partial y^*} + D\beta. \tag{18}$$

Introducing the displacement operator

$$1 + T(x^*; y^*, \eta^*) = \exp\left(a \cdot \frac{\partial}{\partial x^*} + b \cdot \frac{\partial}{\partial y^*} + \beta \cdot \frac{\partial}{\partial \eta^*}\right),$$

We can write (1) - (3) as

$$\frac{dx}{dt} = (1 + T)\ X(x^*; y^*, \eta^*), \tag{19}$$

$$\frac{dy}{dt} = (1 + T)\ \left[\lambda(x^*) + Y(x^*; y^*, \eta^*)\right], \tag{20}$$

$$\frac{d\eta}{dt} = (1 + T)\ H(x^*; y^*, \eta^*). \tag{21}$$

Comparing (16) - (18) with (19) - (21) and changing the notation, we then have

$$\lambda \cdot \frac{\partial a}{\partial y} = (X - X^*) + (TX - Da). \tag{22}$$

$$\lambda \cdot \frac{\partial b}{\partial y} = (Y - Y^*) + T\lambda + (TY - Db), \tag{23}$$

$$\lambda \cdot \frac{\partial \beta}{\partial y} = (H - H^*) + (TH - D\beta). \tag{24}$$

Making use of (14), we can represent $1 + T$ in the form

$$1 + T = \exp \sum_{j=1}^{\infty} \delta_j = \sum_{j=0}^{\infty} T_j, \tag{25}$$

where we put

$$\delta_j = a_j \cdot \frac{\partial}{\partial x} + b_j \cdot \frac{\partial}{\partial y} + \beta_j \cdot \frac{\partial}{\partial \eta}$$

The operators T_j are polynomials in $\delta_1, \delta_2, \cdots$. They can be decomposed into the sums

$$T_j = \sum_{k=1}^{j} T_{j,k}$$

where $T_{j,k}$ are homogeneous and of degree k with respect to the δ-operators. Making use of the expressions obtained by Faa de Bruno (1855) for the higher derivatives of a function depending upon another function, we obtain

$$T_0 = 1$$

$$T_{1,1} = \delta_1$$

$$T_{2,1} = \delta_2$$

$$T_{2,2} = \frac{1}{2}\delta_1^2 ,$$

$$T_{3,1} = \delta_3$$

$$T_{3,2} = \delta_1\delta_2$$

$$T_{3,3} = \frac{1}{6}\delta_1^3$$

$$T_{4,1} = \delta_4$$

$$T_{4,2} = \delta_1\delta_3 + \frac{1}{2}\delta_2^2$$

$$T_{4,3} = \frac{1}{2}\delta_1^2\delta_2$$

$$T_{4,4} = \frac{1}{24}\delta_1^4$$

$$T_{5,1} = \delta_5$$

$$T_{5,2} = \delta_1\delta_4 + \delta_2\delta_3$$

$$T_{5,3} = \frac{1}{2}\delta_1^2\delta_3 + \frac{1}{2}\delta_1\delta_2^2$$

$$T_{5,4} = \frac{1}{6}\delta_1^3\delta_2$$

$$T_{5,5} = \frac{1}{120}\delta_1^5$$

$$T_{6,1} = \delta_6$$

$$T_{6,2} = \delta_1\delta_5 + \delta_2\delta_4 + \frac{1}{2}\delta_3^2$$

$$T_{6,3} = \frac{1}{2}\delta_1^2\delta_4 + \delta_1\delta_2\delta_3 + \frac{1}{6}\delta_2^3$$

$$T_{6,4} = \frac{1}{6}\delta_1^3\delta_3 + \frac{1}{4}\delta_1^2\delta_2^2$$

$$T_{6,5} = \frac{1}{24}\delta_1^4\delta_2$$

$$T_{6,6} = \frac{1}{720}\delta_1^6$$

$$T_{7,1} = \delta_7$$

$$T_{7,2} = \delta_1\delta_6 + \delta_2\delta_5 + \delta_3\delta_4$$

$$T_{7,3} = \frac{1}{2}\delta_1^2\delta_5 + \delta_1\delta_2\delta_4 + \frac{1}{2}\delta_1\delta_3^2 + \frac{1}{2}\delta_2^2\delta_3$$

$$T_{7,4} = \frac{1}{6}\delta_1^3\delta_4 + \frac{1}{2}\delta_1^2\delta_2\delta_3 + \frac{1}{6}\delta_1\delta_2^3$$

$$T_{7,5} = \frac{1}{24}\delta_1^4\delta_3 + \frac{1}{12}\delta_1^3\delta_2^2$$

$$T_{7,6} = \frac{1}{120}\delta_1^5\delta_2$$

$$T_{7,7} = \frac{1}{5040}\delta_1^7$$

$$T_{8,1} = \delta_8$$

$$T_{8,2} = \delta_1\delta_7 + \delta_2\delta_6 + \delta_3\delta_5 + \frac{1}{2}\delta_4^2$$

$$T_{8,3} = \frac{1}{2}\delta_1^2\delta_6 + \delta_1\delta_2\delta_5 + \delta_1\delta_3\delta_4 + \frac{1}{2}\delta_2^2\delta_4 + \frac{1}{2}\delta_2\delta_3^2$$

$$T_{8,4} = \frac{1}{6}\delta_1^3\delta_5 + \frac{1}{2}\delta_1^2\delta_2\delta_4 + \frac{1}{4}\delta_1^2\delta_3^2 + \frac{1}{2}\delta_1\delta_2^2\delta_3 + \frac{1}{24}\delta_2^4$$

$$T_{8,5} = \frac{1}{24}\delta_1^4\delta_4 + \frac{1}{6}\delta_1^3\delta_2\delta_3 + \frac{1}{12}\delta_1^2\delta_2^3$$

$$T_{8,6} = \frac{1}{120}\delta_1^5\delta_3 + \frac{1}{48}\delta_1^4\delta_2^2$$

$$T_{8,7} = \frac{1}{720}\delta_1^6\delta_2$$

$$T_{8,8} = \frac{1}{40320}\delta_1^8$$

The set of operators T_j given here permits one to develop the general perturbations up to the eighth order. The extension of the given table and the check computations can be performed with the general formulas

$$T_{j,k} = \sum \frac{\delta_1^{m1}\ \delta_2^{m2}\ \ldots\ \delta_p^{mp}}{m_1!\ m_2!\ \ldots\ m_p!},$$

$$\sum_{s=1}^{p} m_s = k, \qquad \sum_{s=1}^{p} s m_s = j,$$

$$k T_{j,k} = \sum_{\sigma=1}^{j-k+1} \delta_\sigma T_{j-\sigma,k-1}$$

Taking (15) into account, we can write the operator D as

$$D = \sum_{j=1}^{\infty} D_j, \tag{26}$$

where we put

$$D_j = X_j^* \cdot \frac{\partial}{\partial x} + Y_j^* \cdot \frac{\partial}{\partial y} + H_j^* \cdot \frac{\partial}{\partial \eta}.$$

In order to abbreviate the writing we introduce the symbols

$$L_j \phi = \sum_{k=1}^{j-1} T_{j-k} \phi_k, \quad k = 1, 2, \ldots, j-1$$

$$\Lambda_j \phi = \sum_{k=1}^{j-1} D_{j-k} \phi_k, \quad j = 2, 3, \ldots$$

$$L_1 \phi = 0, \quad \Lambda_1 \phi = 0,$$

representing the result of applying operators $L_j = [T_{j-1}, T_{j-2}, \ldots, T_1, 0, 0, 0 \ldots]$, $\Lambda_j = [D_{j-1}, D_{j-2}, \ldots, D_1, 0, 0, 0 \ldots]$ to the decomposition of ϕ into series with respect to the small parameter.

From (4) - (6), (15), (22) - (24), (25) and (26), we deduce the set of the partial differential equations

$$\lambda \cdot \frac{\partial a_j}{\partial y} = X_j - X_j^* + L_j X - \Lambda_j a, \tag{27}$$

$$\lambda \cdot \frac{\partial b_j}{\partial y} = Y_j - Y_j^* + T_j \lambda + L_j Y - \Lambda_j b, \tag{28}$$

$$\lambda \cdot \frac{\partial \beta_j}{\partial y} = H_j - H_j^* + L_j H - \Lambda_j \beta. \tag{29}$$

For the effects of the first order we have

$$X_1^* = MX_1, \quad Y_1^* = T_1\lambda + MY_1, \quad H_1^* = MH_1,$$

$$\lambda \cdot \frac{\partial a_1}{\partial y} = PX_1, \quad \lambda \cdot \frac{\partial b}{\partial y} = PY_1, \quad \lambda \cdot \frac{\partial \beta_1}{\partial y} = PH_1,$$

and

$$a_1 = QPX_1, \quad b_1 = QPY_1, \quad \beta_1 = QPH_1.$$

Taking into account that the D_j operators contain only the long period terms and that a_j, b_j, β_j contain only the short period terms, we conclude that $\Lambda_j a$, $\Lambda_j b$, $\Lambda_j \beta$ contain no long period terms and thus, to avoid the secular terms in a, b, β, we have to put

$$X_j^* = ML_j X, \tag{30}$$

$$Y_j^* = T_j \lambda + ML_j Y, \tag{31}$$

$$H_j^* = ML_j H \tag{32}$$

It follows from (27) - (29) and (30) - (32):

$$a_j = QP(X_j + L_j X - \Lambda_j a), \tag{33}$$

$$b_j = QP(Y_j + L_j Y - \Lambda_j b), \tag{34}$$

$$\beta_j = QP(H_j + L_j H - \Lambda_j \beta). \tag{35}$$

Evidently (30) - (35) answer the question as to how the long period terms will be formed in higher approximations in the Krylov-Bogoliubov method either directly or as a result of the "cross-action" of the short period effects.

These equations can be written in a somewhat simpler form if the series for X, Y and H are reduced to one term only. Then we have $L_j \phi = T_{j-1} \phi$, and the basic equations become

$$X_j^* = MT_{j-1} X,$$

$$Y_j^* = T_j \lambda + MT_{j-1} Y,$$

$$H_j^* = MT_{j-1} H,$$

and

$$a_j = QP(T_{j-1} X - \Lambda_j a),$$

$$b_j = QP(T_{j-1} Y - \Lambda_j b),$$

$$\beta_j = QP(T_{j-1} H - \Lambda_j \beta).$$

HIGHER ORDER PERTURBATIONS IN THE POINCARÉ AND VON ZEIPEL METHOD

The introduction of the partial differential operators T_j and L_j permits us also to write the equations of the Poincaré and von Zeipel method in a very concise form. Consider the system of canonical equations:

$$\frac{dx}{dt} = + \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = - \frac{\partial F}{\partial x},$$

$$\frac{d\xi}{dt} = + \frac{\partial F}{\partial \eta}, \quad \frac{d\eta}{dt} = - \frac{\partial F}{\partial \xi}.$$

We assume that the Hamiltonian F is developable in powers of a small parameter and has the form

$$F = F_0(x) + F_1(x, \xi; y, \eta) + F_2(x, \xi; y, \eta) + \cdots \tag{36}$$

The functions F_j $(j = 1, 2, 3, \cdots)$ are periodic in y and η with period 2π in each component:

$$F_j = \sum_{n, \nu} F_{j,n,\nu}(x, \xi) \exp i \, (n \cdot y + \nu \cdot \eta).$$

We shall determine a canonical transformation

$$x = x^* + \frac{\partial S}{\partial y}, \quad y^* = y + \frac{\partial S}{\partial x^*},$$

$$\xi = \xi^* + \frac{\partial S}{\partial \eta}, \quad \eta^* = \eta + \frac{\partial S}{\partial \xi^*},$$

$$S(x^*, \xi^*; y, \eta) = \sum_{j=1}^{\infty} S_j(x^*, \xi^*; y, \eta)$$

such that the new Hamiltonian

$$F^* = F_0^* + F_1^* + F_2^* + \cdots \tag{37}$$

does not contain the short period argument y^*; in other words, such that the condition

$$F(x, \xi; y, \eta) = F^*(x^*, \xi^*; -, \eta^*) \tag{38}$$

is satisfied.

Putting

$$h(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial y},$$

$$k(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial x^*},$$

$$\chi(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial \eta},$$

$$\kappa(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial \xi^*}$$

we write (38) as

$$F\left[x^* + h(x^*, \xi^*; y, \eta), \xi^* + \chi(x^*, \xi^*; y, \eta); y, \eta\right] = F^*\left[x^*, \xi^*; -, \eta + \kappa(x^*, \xi^*; y, \eta)\right],$$

or, changing the notation, as

$$F\left[x + h(x, \xi, y, \eta), \xi + \chi(x, \xi; y, \eta); y, \eta\right] = F^*\left[x, \xi; -, \eta + \kappa(x, \xi; y, \eta)\right]. \tag{39}$$

Introducing the displacement operators

$$1 + T(x, \xi; y, \eta) = \exp\left[h(x, \xi; y, \eta)\cdot\frac{\partial}{\partial x} + \chi(x, \xi; y, \eta)\cdot\frac{\partial}{\partial \xi}\right],$$

and

$$1 + T^*(x, \xi; y, \eta) = \exp \kappa(x, \xi; y, \eta)\cdot\frac{\partial}{\partial \eta},$$

we can write (39) in the form

$$\left[1 + T(x,\ \xi;\ y,\ \eta)\right]\ F(x,\ \xi;\ y,\ \eta) = \left[1 + T^*(x,\ \xi;\ y,\ \eta)\right]\ F^*(x,\ \xi;\ -,\ \eta). \tag{40}$$

Let us define the operators δ_k and δ_k^* by means of the equations

$$\delta_k = \frac{\partial S_k}{\partial y} \cdot \frac{\partial}{\partial x} + \frac{\partial S_k}{\partial \eta} \cdot \frac{\partial}{\partial \xi},$$

$$\delta_k^* = \frac{\partial S_k}{\partial \xi} \cdot \frac{\partial}{\partial \eta}.$$

Then we have, similarly as before,

$$1 + T = \exp \sum_{j=1}^{\infty} \delta_j = \sum_{j=0}^{\infty} T_j, \tag{41}$$

$$1 + T^* = \exp \sum_{j=1}^{\infty} \delta_j^* = \sum_{j=0}^{\infty} T_j^*, \tag{42}$$

$$L_j = \left[T_{j-1},\ T_{j-2}, \ldots,\ T_1.\ 0,\ 0, \ldots\right]$$

$$L_j^* = \left[T_{j-1}^*,\ T_{j-2}^*, \ldots,\ T_1^*,\ 0,\ 0, \ldots\right].$$

The operators T_j are expressible in terms of δ_k, and the operators T_j^* in terms of δ_k^*, by means of the formulas given in the previous section. Making use of (37), (38), (41), and (42), we obtain

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j} T_{j-k} F_k = \sum_{j=0}^{\infty} \sum_{k=0}^{j} T_{j-k}^* F_k^*,$$

$$F_0^* = F_0(x),$$

$$T_j^* F_0^* = 0,\ j > 0,$$

and consequently

$$\sum_{k=0}^{j} T_{j-k} F_k = \sum_{k=1}^{j} T_{j-k}^* F_k^* \qquad \begin{cases} j = 1,\ 2,\ 3,\ \ldots \\ k = 0,\ 1,\ \ldots,\ j \end{cases} \tag{43}$$

Taking into account

$$T_j^* F_0 = \frac{\partial S_j}{\partial y} \cdot \frac{\partial F_0}{\partial x} + (T_j - \delta_j) F_0,$$

we can rewrite (43) as

$$\lambda \cdot \frac{\partial S_j}{\partial y} + \Phi_j = F_j^*, \tag{44}$$

where

$$\lambda = \frac{\partial F_0}{\partial x}$$

and

$$\Phi_j = F_j + (T_j - \delta_j) F_0 + L_j F - L_j^* F^*.$$

From the system of the linear partial differential equations (44) we can determine the functions S_j and F_j^* step by step. In order to dispose of the secular terms in S_j we have to put $F_j^* = M \Phi_j$; then we obtain

$$S_j = OP\Phi_j.$$

The system of transformed equations becomes

$$\frac{dx^*}{dt} = + \frac{\partial F^*}{\partial y^*} = 0, \quad \frac{dy^*}{dt} = - \frac{\partial F^*}{\partial x^*} \tag{45}$$

$$\frac{d\xi^*}{dt} = + \frac{\partial F^*}{\partial \eta^*}, \quad \frac{d\eta^*}{dt} = - \frac{\partial F^*}{\partial \xi^*}. \tag{46}$$

Besides the integral of energy, the new system also possesses the integral $x^* =$ const.

The system (46) can be integrated independently from the system (45) and after the integration the angle y^* can be obtained by a plain quadrature.

A further reduction is possible if F^* can be re-arranged so that the purely secular term is of lower order than the periodic terms. By repeating the process of elimination of the periodic terms, we can obtain the solution of the original problem in the form of a Fourier series with arguments linear with respect to time. Such a reduction is possible in the case of the artificial satellite of the earth (Brouwer, 1959), but it is not always possible in the lunar problem or in the stellar three body problem. If the close companion (the lunar orbiter) is in a highly eccentric orbit and the osculating plane has a high inclination toward the orbital plane of the distant companion (Brown, 1936, 1937), then the solution in the form of standard trigonometric series generally speaking cannot be obtained.

CONCLUSION

The method of Krylov and Bogoliubov does not presuppose that the forces are conservative. Thus, the importance and the generality of this method are quite evident. The system of the differential operators and the algorithm given here permit the computation of the higher order effects up to any order. The process is formal and from the standpoint of pure mathematics might suffer, as all astronomical theories do, from the presence of small divisors.

Recently, the method of Krylov and Bogoliubov was successfully applied by Struble (1961) and by Kyner (1965) to the problem of motion of the artificial satellite. The author of the present paper has applied it to the problem of the motion of a lunar orbiter, and an exposition of the results will appear in a later paper.

REFERENCES

Bogoliubov, N., and Mitropolsky, Y. A., "Asymptotic Methods in the Theory of Non-Linear Oscillations," 2nd ed. (Translated from Russian), New York: Gordon and Breach, 1961.

Brouwer, D., "Solution of the Problem of Artificial Satellite Theory without Drag," *Astron. J.* 64(9):378-397, November 1959.

Brown, E. W., "Stellar Problems of Three Bodies. I: Application of Satellite Theory," *Roy. Astron. Soc. Month. Not.* 97(1):56-61, November 1936.

Brown, E. W., "The Stellar Problem of Three Bodies. II: The Equations of Motion with a Simplified Solution," *Roy. Astron. Soc. Month. Not.* 97(1):62-66, November 1936.

Brown, E. W., "The Stellar Problem of Three Bodies. III: The Motions of the Apse and Node with Applications to the Moon," *Roy. Astron. Soc. Month. Not.* 97(2):116-127, December 1936.

Brown, E. W., "The Stellar Problem of Three Bodies. IV: Perturbations in the System & Ursae Majoris," *Roy. Astron. Soc. Month. Not.* 97(5):388-395, March 1937.

Faà De Bruno, M., "Note sur une Nouvelle Formule de Calcul Differentiel," *Quart. J. Math.* 1(4): 359-360, 1857.

Giacaglia, G., "Notes on von-Zeipel's Method," Goddard Space Flight Center document X-547-64-161, 1964.

Kyner, W. T., "A Mathematical Theory of the Orbit about an Oblate Planet," *J. Soc. Industr. and Appl. Math.* 13(1):136-174, March 1965.

Leverrier, U. J., "Remarques . . . sur les Instruments Magnétiques Récémment Etablis à L'Observatoire de Paris," In: *Annuaire de la Société Météorologique de France* IV:73, April 8, 1956.

Poincaré, H., "Les Méthodes Nouvelles de la Méchanique Céleste. Méthodes de MM. Newcomb, Gyldén, Lindstedt, et Bohlin," Vol. II, Paris: Gauthier-Villars et fils, 1893.

Struble, R. A., "The Geometry of the Orbits of Artificial Satellites," *Arch. Rational Mech. Anal.* 7:87-104, 1961.

von Zeipel, H., "Recherches sur le Mouvement des Petites Planètes," Arkiv Matem. Astron. o. Fysik, 11(1):1-58, 1916-1917.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546